Vortices near surfaces of Bose-Einstein condensates

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The theory of vortex motion in a dilute superfluid of inhomogeneous density demands a boundary layer approach, in which different approximation schemes are employed close to and far from the vortex, and their results matched smoothly together. The most difficult part of this procedure is the hydrodynamic problem of the velocity field many healing lengths away from the vortex core. This paper derives and exploits an exact solution of this problem in the two-dimensional case of a linear trapping potential, which is an idealization of the surface region of a large condensate. It thereby shows that vortices in inhomogeneous clouds are effectively 'dressed' by a non-trivial distortion of their flow fields; that image vortices are not relevant to Thomas-Fermi surfaces; and that for condensates large compared to their surface depths, the energetic barrier to vortex penetration disappears at the Landau critical velocity for surface modes.

INTRODUCTION

Although much has long been known about quantized vortices in superfluids of constant homogeneous density, vortices in trapped dilute Bose-Einstein condensates move and interact within significantly inhomogeneous background clouds. The motion of vortices in an inhomogeneous superfluid, including their stability and location in a rotating condensate, has been the subject of several recent theoretical works, both numerical [1, 2, 3, 4, 5, 6] and analytic [7, 8, 9, 10, 11, 12, 13, 14]. After years of effort, vortices in dilute condensates are now also a subject of active experimental study [15, 16, 17, 18, 19, 20, 21, 22]. Quantitative comparison between theory and experiment therefore requires refinement of theoretical methods, in order to go beyond qualitative or order-of-magnitude estimates, and derive more rigorous predictions.

This paper will take a step in this direction, by providing an exact solution (in a non-trivial but two-dimensional case) to a part of the problem that has usually been treated with uncontrolled approximations. Our specific conclusions will include the verdicts that image vortices are inapplicable to surface effects in experimental condensates, and that the often-cited energetic barrier to vortex penetration [3] disappears at velocities no greater than the surface mode critical velocity [23, 24], well below previous estimates [1, 13]. This conclusion applies in the experimentally relevant limit of a condensate large compared to its surface depth, and the lowering of critical rotation frequencies which it implies is an addition to any effects of asymmetric potentials [14]. The case we examine also provides a general warning against neglecting the nontrivial hydrodynamics of inhomogeneous superfluids.

Boundary layer theory

In a dilute superfluid whose macroscopic wavefunction Ψ is governed by the Gross-Pitaevskii equation

$$i\hbar\partial_t\Psi = -\frac{\hbar^2}{2M}\nabla^2\Psi + V(\vec{x})\Psi + \frac{\hbar^2 g}{M}|\Psi|^2\Psi , \qquad (1)$$

the core diameter of a vortex at position $\vec{x}_0$ is on the order of the local *healing length* $\xi = [g\rho(\vec{x}_0)]^{-1/2}$, where M is the particle mass, g is the interaction strength, and $\rho \equiv |\Psi|^2$ is the condensate number density, which varies smoothly over the sample from zero to its maximum value. This maximum is fixed by the chemical potential μ, which is not a parameter in the Gross-Pitaevskii equation, but an integration constant: the general solution admits a time-dependent prefactor in Ψ of the form $e^{-i\mu t/\hbar}$.

Throughout this paper we will consider a quasi-two-dimensional condensate, so that g is dimensionless [26]. Our results may also be applied, with trivial changes, to the case of a hydrodynamically 3D condensate in which the density is constant along the third direction, and the vortex lines are parallel to it. In three-dimensional background clouds, however, vortex lines generally bend [8, 13, 25], and this introduces complications which usually prohibit accurate analytical treatment [8]; hence our restriction, in this paper, to the two-dimensional case. Even in the limit of a quasi-two-dimensional condensate, however, the inhomogeneous background density of a trapped condensate significantly affects the behaviour of two-dimensional 'point vortices'. The Thomas-Fermi (TF) surface, which is the edge of the condensate cloud in the hydrodynamic approximation [27], also affects vortex motion.

It is typically the case, in current experiments, that near the vortex the trapping potential V varies slowly on the healing length scale. This small ratio of scales has two distinct implications for vortex physics. Firstly, it means that within many healing lengths around the vortex center, the gradient in V can be treated as a perturbation. Secondly, it means that beyond a few healing

lengths from the vortex center, we may solve (1) in the *hydrodynamic approximation.* These are two very different kinds of approximation, but neither is valid everywhere, and so they must be combined. Fortunately there is a significant intermediate region near the vortex within which both approximations are valid, and this will allow us to match the solutions they yield smoothly together, to obtain a complete solution. This is an example of a general procedure, known as boundary layer theory, or the method of matched asymptotics.

Boundary layer theory is more than just a technical trick without physical meaning: it is an essential feature of their physics that vortices are small, healing-length-scale structures, realized within a larger-scale hydrodynamic medium. Boundary layer theory is a very direct expression of the multiple-scale nature of vortices, and the only alternatives to it are exact (or numerical) solutions, which will of course also exhibit the problem's multiple scales, or the replacement of some part of the boundary layer analysis with some uncontrolled approximation. The boundary layer method has yielded general equations of motion for dark solitons in quasi-one-dimensional inhomogeneous backgrounds [28], and has already been applied to 2D vortex motion in inhomogeneous backgrounds [7, 13]. In this paper we will take a precisely similar approach. Our main contribution is an exact solution to the outer, hydrodynamic part of the problem, in a physically relevant and nontrivial case. Comparing our exact solution to the assumptions of previous work will illustrate some general aspects of vortex physics that may not have been fully appreciated heretofore.

Organization

This paper is organized as follows. In the following Section II we review the hydrodynamic approximation, and then identify a loophole in previous derivations of a general solution supposed to be valid in the neighbourhood of a vortex. We show that in fact a nontrivial problem must be solved, which generically lacks small parameters and involves the whole condensate globally. We then present a physically relevant case in which this problem can be solved exactly, in terms of a special function which is only moderately obscure, and whose asymptotic behaviours close to and far from the vortex can be obtained analytically.

In Section III we follow earlier authors [7] in obtaining the inner solution, perturbing around the numerical solution to the Gross-Pitaevskii equation for a vortex in constant background density. We then go on to match the inner and outer solutions together, and thereby determine the velocity at which the vortex moves parallel to the TF surface. With these results, we compute the free energy of a vortex in a moving frame, and thus assess

the velocity at which the energetic barrier preventing vortices from entering the condensate will disappear. We compare these results to recent analogous calculations for critical rotation frequencies of harmonically trapped condensates, concluding that the latter overestimate critical frequencies of large condensates by factors of order unity.

In our final Section IV we discuss our results, interpreting the vortex motion as due to vortex buoancy, through a Magnus effect which is renormalized by the distortion of the flow field. We interpret this distortion as an 'infrared dressing' of the vortex, and emphasize that such dressing is a much more general effect than can be described with image vortices. We argue that the energetic barrier to vortices actually disappears at or below the surface mode critical velocity [23, 24], so that, strictly speaking, vortices entering condensates should not be said to 'nucleate' (cross an energy barrier by thermal fluctuations or quantum tunneling). We then conclude with a brief summary and outlook.

OUTER SOLUTION: HYDRODYNAMICS

Hydrodynamic approximation

The hydrodynamic approximation works as follows. If we define $\Psi = \sqrt{\rho}e^{i\theta}$ for real ρ, θ, the Gross-Pitaevskii equation becomes exactly

$$\partial_t \rho = -\frac{\hbar}{M}\vec{\nabla}\cdot\rho\vec{\nabla}\theta \quad \text{(continuity)} \tag{2}$$

$$\partial_t \theta = -\frac{\hbar}{M}\left[\frac{1}{2}|\vec{\nabla}\theta|^2 + \frac{M}{\hbar^2}V + \frac{1}{\xi^2}\frac{\rho}{\rho_0} - \frac{\nabla^2\sqrt{\rho}}{2\sqrt{\rho}}\right] \tag{3}$$

$$\rho_0 \equiv \rho(\vec{x}_0),$$

where the vortex center is located at $\vec{x}_0$. We write g in the form $1/(\xi^2\rho_0)$ in order to introduce the vortex scale ξ explicitly. We seek a solution in which the only time dependence is due to the vortex motion, so that sufficiently far from $\vec{x}_0$ we can set $\dot{\rho} = 0$, and $\dot{\theta} = -\mu/\hbar$ with constant μ. If θ and V are functions of $\varepsilon\vec{x}/\xi$ for some small ε, then the second equation (3) just yields the Thomas-Fermi density profile $\rho_{TF}(\vec{x}) = M(\mu - V)/(\hbar^2 g)$, up to corrections of order ε or higher. The surface on which ρ_{TF} vanishes is the TF surface, where the TF approximation breaks down. (An additional boundary layer treatment is therefore required very close to the TF surface [27].)

In this case we may expect (and later confirm) that the vortex velocity v_{vtx} is order εc, where $c = \hbar/(M\xi)$ is the speed of bulk sound near the vortex. This means that the corrections are formally of order ε^2. But there will also be corrections that diverge as one approaches the vortex, the leading one being the vortex kinetic energy $\propto r^{-2}$. To keep these corrections no larger than order ε,

we can only use the hydrodynamic approximation in an 'outer zone' whose inner boundary is a circle around the vortex centre, having radius R of order $\varepsilon^{-1/2}\xi$. In the 'inner zone' inside this radius, we will use perturbation theory on the potential gradient instead. As long as it has the right order of magnitude, the precise value of R is arbitrary: it is merely a bookkeeping device to let us merge two different approximations, and all R-dependent terms necessarily cancel when the two zones are patched together. So the full calculation will in fact be accurate up to corrections of order ε^2 after all. And this definition of the outer zone also formalizes the requirement of being 'sufficiently far' from the vortex to set $\dot{\rho} = 0$ and $\dot{\theta} = -\mu/\hbar$, since for $|\vec{x} - \vec{x}_0| > R$ the corrections $\vec{v}_{vtx} \cdot \vec{\nabla}\rho, \theta$ will be of order ε^2.

The higher order corrections can all be computed trivially once the phase field θ is known to zeroth order. (We will not actually compute such corrections in this paper.) The zeroth order phase field can be obtained by solving the continuity equation (2) with the zeroth order TF density, and so this is the main problem of the outer zone.

Dual fields

Because the phase will be multi-valued in the presence of vortices, it is convenient in two dimensions to define the field $\vec{A}$ which is dual to the velocity field (in the sense in which electric and magnetic fields may be dual):

$$(A_x, A_y) \equiv (\partial_y\theta, \ -\partial_x\theta) \ . \tag{4}$$

The continuity equation then becomes $\vec{\nabla} \times (\rho\vec{A}) = 0$, which can be identically satisfied by setting

$$\vec{A} \equiv \frac{\rho_0}{\rho}\vec{\nabla}F \tag{5}$$

for some potential $F(x, y)$. We will refer to F as the dual potential to the phase θ, but it is also known as the streamline function, because its contours of constant height are streamlines of the velocity field. Because it is single-valued even when vortices are present, F is a more convenient substitute for θ. F is not arbitrary, because we can see from (4) that $\vec{\nabla} \cdot \vec{A} = \vec{\nabla} \times \vec{\nabla}\theta = 2\pi\,\delta^2(\vec{x} - \vec{x}_0)$ (since a singly quantized vortex is located at $\vec{x}_0$). Hence F must satisfy $\rho_0\vec{\nabla}\cdot\left(\rho^{-1}\vec{\nabla}F\right) = 2\pi\,\delta^2(\vec{x} - \vec{x}_0)$. But since we are only seeking the zeroth order phase field, we can replace $\rho \to \rho_{TF}$ in both (5) and in the constraint on F, writing $\vec{A} \doteq (\rho_0/\rho_{TF})\vec{\nabla}F$ as well as

$$\rho_0\vec{\nabla}\cdot\frac{\vec{\nabla}F}{\rho_{TF}} \doteq 2\pi\,\delta^2(\vec{x} - \vec{x}_0) \tag{6}$$

where the $\doteq$ means the neglect of terms that will ultimately lead to corrections of order ε^2. (Introducing $_0$

subscripts on F and $\vec{A}$ would too greatly encumber our notation.) Eqn. (6) is the essential equation determining the zeroth order outer zone phase field of a vortex. Note that since it is a linear equation, solutions with more vortices can be obtained trivially from the single vortex case.

As a simple example, consider the classic case of a vortex in a constant background density profile, near a hard wall. Choose the y axis to run along the wall, and let the vortex be located on the x axis at the point x_0. Since in this case $\rho/\rho_0 = 1$, we merely have a Laplace equation to solve, with a delta-function source, and the boundary condition that there be no flow through the line $x = 0$. Since adding a constant to F obviously does nothing, we can enforce this boundary condition by setting $F(0, y) = 0$. This is satisfied by the solution

$$F_{HW} = \frac{1}{2}\ln\frac{(x - x_0)^2 + y^2}{(x + x_0)^2 + y^2}$$

where a possible extra term proportional to x must vanish in the case that the velocity field vanishes at infinity. This solution can evidently be obtained by the method of images, with the denominator of the logarithm representing the dual potential of an image antivortex located at $-x_0$.

Obtaining an analogous solution for an inhomogeneous ρ is generally much more difficult, however.

Proposed general results

General equations of motion for point vortices in inhomogeneous superfluid backgrounds have nevertheless been presented, having been derived using the boundary layer approach [7, 11]. According to these proposed equations, vortex motion is determined by the local trapping potential in the immediate neighbourhood of the vortex (that is, can be given by a universal formula involving the potential, its gradient, and its Laplacian at $\vec{x}_0$). We pause here to point out a loophole in the derivation of these results, which unfortunately allows corrections to vortex motion that are comparable in size to the proposed general results, and that in general cannot be computed without solving the global hydrodynamic problem.

The approach of Rubinstein and Pismen, and of Svidzinsky and Fetter following them, is the same approach we are following, of matching a hydrodynamic outer zone with a perturbative inner zone. At the point we have now reached, namely solving Eqn. (6), these authors employ the variable $H(x, y) = \sqrt{\rho_0/\rho}\,F(x, y)$ (Ref. [11] uses a more elaborate notation for the same function) and so obtain the equivalent equation,

$$\left[\nabla^2 - k^2\right]H \ = \ 2\pi\,\delta(x - x_0)\,\delta(y) \tag{7}$$

$$\text{for } \ k^2 \ \equiv \ \frac{1}{2}\left[-\frac{\nabla^2\rho}{\rho} + \frac{3}{2}\left|\frac{\vec{\nabla}\rho}{\rho}\right|^2\right].$$

Their general procedure, for any slowly varying potential, is then to replace $k(x) \rightarrow k(x_0) = k_0$, so that (7) becomes simply a 2D Helmholtz equation (with a delta-function source). It is then observed that the modified Bessel function $H(x,y) = -K_0\left(k_0\sqrt{(x-x_0)^2 + (y-y_0)^2}\right)$ is a particular solution to (7) with $k(x) \rightarrow k_0$. Of course only the $(x,y) \rightarrow (x_0,y_0)$ limit of this result is to be taken seriously, so the Rubinstein-Pismen result for the outer solution near the vortex is

$$F_{RP}\left(x_0 + \xi r\cos\phi \ , \ y_0 + \xi r\sin\phi\right)$$
$$= \ \sqrt{1 + \xi\rho_0^{-1}\vec{r}\cdot\vec{\nabla}\rho_0}\left(\gamma + \ln\frac{r\xi k_0}{2}\right) + \mathcal{O}(\varepsilon^2).$$

Euler's constant $\gamma \doteq 0.577$ appears through the asymptotic behaviour of the modified Bessel function at small argument.

The problem with this computation is that it assumes that the solution for H contains no components regular at $(x,y) \rightarrow (x_0,y_0)$ (terms like $I_0, I_1\cos\phi$, etc.). Actually, nothing ensures this. And so all that can really be claimed, on the basis of purely local analysis near the vortex, is that

$$\lim_{\vec{x}\to\vec{x}_0} F \ = \ \left(1 + \frac{\xi}{2}\vec{r}\cdot\vec{\nabla}\rho_0\right)\ln r$$
$$+ k_0\xi r(B_y\cos\phi - B_x\sin\phi) + \mathcal{O}(\varepsilon^2) \quad (8)$$

for some undetermined constants $B_{x,y}$. (A constant term can be dropped because it will not contribute to the dual velocity field $\vec{A}$.) This unknown $\vec{B}$ implies an unknown correction to the superfluid velocity field near the vortex, and hence to the vortex velocity. So the strictly local approach, based on solving (7) with $k(\vec{x}) \rightarrow k_0$, does not actually lead to any conclusion at all for the vortex motion.

In Reference [7], Rubinstein and Pismen are careful to state that they are deriving the vortex velocity relative to the 'ambient' superfluid velocity, thus recognizing the corrections they are omitting. It is worth emphasizing, however, that the fluid velocity that may be called 'ambient' is in a sense scale dependent. The effect omitted in Refs. [7, 11] is indeed a flow near the vortex that is constant *on the healing length scale*; but it is generally *not* constant on the much longer hydrodynamic scale of the outer zone. So while such a flow may be called 'ambient' from the point of view of the vortex core, in general it is *not* simply the flow that is specified at infinity (or wherever boundary conditions are imposed).

Whereas Rubinstein and Pismen recognize but omit them, Svidzinsky and Fetter argue in Reference [11] that additional terms regular at the vortex may be ruled out, because the I_n functions all diverge at large argument, and so would violate any physical boundary conditions. While it is true that the I_n all diverge at large argument, they only do this on the scale k_0, and once $k_0|\vec{x} - \vec{x}_0|$ is no longer small, the approximation $k(\vec{x}) \rightarrow k_0$ is no longer good. Hence the argument of [11] is invalid.

Motivation for an exact solution

The moral of this general discussion should be clear. Implementing the hydrodynamic approximation has exhausted the benefits of the slow variation of the trapping potential on the healing length scale and thus, in general, Eqn. (6) contains no small parameters. So except in special cases where additional small parameters may appear, such as for a trap with a high aspect ratio, or a vortex very close to the center of symmetric trap, no accurate equation of motion for a vortex may be obtained without solving a nontrivial hydrodynamic problem exactly.

In [7], Rubinstein and Pismen provide one nontrivial exact solution, for the case described in our variables as $M[\mu - V(\vec{x})] = \hbar^2 g\rho_0[I_0(r/\mathcal{R})]^{-2}$ for some $\mathcal{R} \gg \xi$. They focus on this case because it ensures that $k(x)$ in (7) really is constant, so that their solution quoted above becomes exact. The trap required to realize their solvable example with trapped dilute condensates is not implausible (it closely resembles a Gaussian well); but one would also need a finely tuned chemical potential, to make the condensate density just vanish at infinity. And it is a special feature of this finely-tuned case that there is no Thomas-Fermi radius. (Corrections to the Thomas-Fermi profile begin to exceed $(\xi/\mathcal{R})^2$ for $r > \mathcal{R}\ln(\mathcal{R}/\xi)$, but their onset is very gradual.) It is therefore the main contribution of this paper to provide, in the next subsection, another instructive exact solution to the hydrodynamic problem of a superfluid vortex in an inhomogeneous background, which is more directly relevant to currently typical experiments.

Vortex hydrodynamics in a linear density profile

The plane linear approximation

Near a Thomas-Fermi surface, the potential is approximately linear; and the TF surface of a large condensate is approximately flat. This motivates considering the idealized problem of a linear ramp potential [23, 24, 27, 29]. Choosing the y axis to run along the straight TF surface, we have the TF profile

$$\rho = \rho_0\frac{x}{x_0}. \quad (9)$$

Since it will turn out that F decays on the distance scale of x_0, the linear ramp potential will indeed be reasonably accurate for real traps, which are obviously not globally linear, as long as the vortices are not too far from the TF surface; for harmonic traps, this means that x_0 should be much less than the TF radius. On the other hand, with $V = \mu - \lambda x$ (obtained automatically by taking $x = 0$ on the TF surface), we have $\xi = \hbar/\sqrt{M\lambda x_0}$, and hence $\varepsilon = \xi/x_0 = \hbar/\sqrt{M\lambda x_0^3}$. The hydrodynamic approximation breaks down within a few surface depths of the TF surface, where the surface depth [27] is

$$\delta = \left(\hbar^2/2M\lambda\right)^{1/3}. \tag{10}$$

We therefore also have $\varepsilon = \sqrt{2}\left(\delta/x_0\right)^{3/2}$, and so our boundary layer treatment of the vortex will require $x_0 \gg \delta$. See Figure 1 for a sketch of our model system indicating the relationships between the various length scales that will appear in our calculations.

Vortices located well outside the TF surface ($x_0 \ll -\delta$) can be described perturbatively as surface excitations of the condensate [24]; but vortices with centers within the surface layer $|x| \lesssim \delta$ require a nonperturbative treatment, which we are unable to provide. In our final Section, though, we will argue that nothing remarkable can happen in this regime, which is analytically intractable but physically trivial. As mentioned above, the TF surface requires a boundary layer treatment of its own [27], independent of the vortex; but the post-hydrodynamic effects within this layer can easily be shown to produce corrections smaller than all results we will report by factors of at least $(\delta/x_0)^2$. Hence for vortices many surface depths into the condensate, we will be able to ignore completely the breakdown of hydrodynamics at the TF surface itself.

The equation

Setting $y_0 = 0$ by our choice of origin, Eqn. (6) in this case becomes

$$[\partial_{xx} - \frac{1}{x}\partial_x + \partial_{yy}]F = 2\pi\delta(x - x_0)\,\delta(y). \tag{11}$$

As warned above, this equation has no small parameters in it: even x_0 may be scaled away. Fortunately, however, (11) is exactly solvable. Taking advantage of translational symmetry in y, we can write

$$F(x, y) = \int_{-\infty}^{\infty} dk\, f_k(x)\, e^{iky} \tag{12}$$

to obtain the ODE

$$f_k'' - \frac{1}{x}f_k' - k^2 f_k = \delta(x - x_0)\,. \tag{13}$$
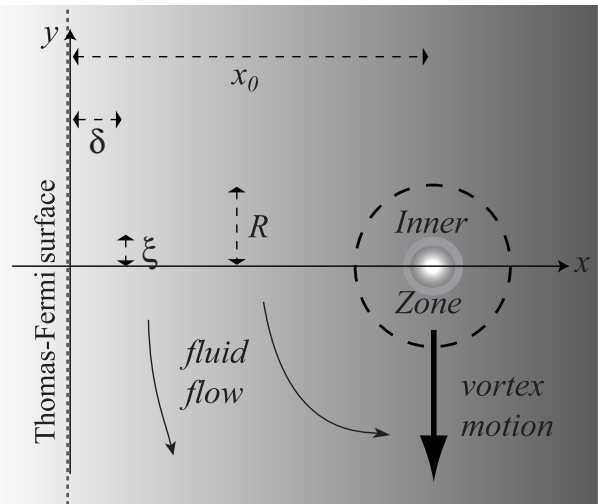

FIG. 1: Sketch with scales. Darkness of the background indicates condensate density, increasing linearly with distance x from the Thomas-Fermi surface at left, on which the y axis is placed. The hydrodynamic approximation is used in the Outer Zone, further than R from the vortex center; in the Inner Zone the full Gross-Pitaevskii equation is solved with the potential gradient treated as a perturbation. The relative sizes of the four scales ξ, δ, R and x_0 are indicated qualitatively, but the ratios should obviously be rather greater for the analysis in the text to be accurate. The counter-clockwise circulating vortex is shown in the text to move in the negative y direction.

We begin by solving the homogeneous equation, setting the RHS to zero. Multiplying f_k by x allows us to recognize a modified Bessel equation, and so we obtain solutions involving modified Bessel functions of order 1:

$$f_k^0 = x\left[A_k K_1(kx) + B_k I_1(kx)\right] \tag{14}$$

for constants A_k, B_k. We wish to impose the hydrodynamic boundary condition of no velocity through the TF surface, corresponding to $F(0, y) = $ a constant that we can choose to be zero; so since $\lim_{x\to 0} K_1(x) = 1/x$, we need $A_k = 0$. We can accept no solutions that grow exponentially as $x \to \infty$, and so we must also set all $B_k = 0$. The only exception that might occur is the special case $k = 0$, in which we might under some circumstances require $f_0^0 = B_0 x^2$, since $F \propto x^2$ corresponds merely to a constant velocity field in the y direction. But for the case where the velocity vanishes at infinity, we need $B_0 = 0$ as well.

Note the difference here between our rejection of additional homogeneous solutions, and their rejection in [11]: we have solved our hydrodynamic problem exactly for all x, and not just asymptotically near the vortex. Of course, to apply our model to a realistic case we must admit that the potential becomes nonlinear at sufficiently large x or y; but as long as this nonlinearity scale (call it Λ) is much larger than x_0, our exact solution will be applicable over an essentially infinite region, as far as the

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vortex is concerned. One loophole does remain, however, even in this case. We have fixed $B_0 = 0$ by demanding that the velocity field vanish at infinity, to which we assume that our linear model of the potential extends. But in a real, finite system, even if it is modelled well by a linear potential over a large region, it could be that boundary conditions farther away from the vortex than Λ imply some nontrivial, nonzero B_0. This would allow corrections to the vortex velocity that one can expect to be of order $\hbar/(\Lambda M)$. To compute them would require solving the actual, finite hydrodynamic problem.
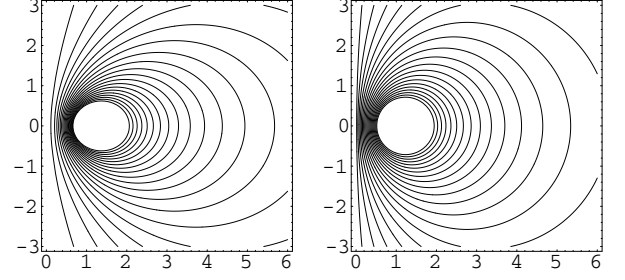

FIG. 2: Hydrodynamic flow lines around a 2D point vortex near (left) a TF surface and (right) a hard wall. In both cases the surface of the condensate is the vertical axis, with the axes marked in units of x_0. The left plot is constant height contours of $F = -\sqrt{x/x_0}Q_{1/2}(z)$ for $z(x,y)$ defined in the text. The right plot consists of constant height contours of the homogeneous dual potential with a hard wall surface, $F_{HW} = \ln[\sqrt{(x-x_0)^2+y^2}/\sqrt{(x+x_0)^2+y^2}]$,.

The vortex solution

Having solved our homogeneous equation in general and concluded that the only solution admitted by our boundary conditions is zero, there remains the main task of obtaining a particular solution for F with the delta function source at $x = x_0$. Since we have the general solutions to the homogeneous equation, we can patch them together to meet our boundary conditions and also fit the delta function source, by writing

$$f_k = C\frac{x}{x_0} \times \left\{ \begin{array}{l} K_1(kx_0)I_1(kx) \ , \ x < x_0 \\ I_1(kx_0)K_1(kx) \ , \ x > x_0 \end{array} \right\} \quad (15)$$

for a constant C. We then fix C by imposing that the discontinuity in the derivative of f_k at $x = x_0$ be equal to 1, as required by (13). Since $K_1'(\xi)I_1(\xi) - K_1(\xi)I_1'(\xi) = -1/\xi$, this gives $C = -1$, and so we have

$$F = -2\frac{x}{x_0}\int_0^\infty dk \cos ky \, [K_1(kx_0)I_1(kx)\theta(x_0-x) + K_1(kx)I_1(kx_0)\theta(x-x_0)] \quad (16)$$

where $\theta(x)$ is the step function. Happily, this integral can be evaluated in closed form in terms of a Legendre function of the second kind [30], yielding

$$F(x,y) = -\sqrt{x/x_0}Q_{\frac{1}{2}}(z) \ , \quad (17)$$

where $z = z(\frac{x}{x_0}, \frac{y}{x_0})$ is defined as

$$z = \frac{x^2+y^2+x_0^2}{2xx_0} = 1 + \frac{(x-x_0)^2+y^2}{2xx_0} \quad (18)$$

The Legendre functions of order ν satisfy the Legendre differential equation

$$\frac{d}{dz}\left[(1-z^2)\frac{dQ_\nu}{dz}\right] = -\nu(\nu+1)Q_\nu \quad (19)$$

and the functions of the second kind have logarithmic singularities at $z = \pm 1$. It is straightforward to verify, using (19), that (17) satisfies (6) with the boundary condition $F(0,y) = 0$.

The vortex flow field can be visualized by plotting contour lines of constant F which correspond to fluid flow lines. Such contour plots are available numerically through common commercial software applications that include large libraries of special functions. An example is shown in Figure 1, with the corresponding flow pattern for a vortex in homogeneous density near a hard wall, for comparison. It is obvious that the density gradient distorts the flow pattern signficantly.

Outer asymptotics

At large z, $Q_{1/2} \to 2^{-5/2}\pi z^{-3/2}$ (easily obtained from (25) below, or see [31]), so that the dual potential F falls off at long range much more quickly than the long-range logarithmic behaviour found at constant background density. Inserting our definition of z we therefore have

$$\lim_{|y|\to\infty} F(x,y) = -\frac{\pi x^2 x_0}{2|y|^3}\left(1+\frac{x^2}{y^2}\right)^{-3/2} \quad (20)$$

$$\lim_{x\to\infty} F(x,y) = -\frac{\pi x_0}{2x}\left(1+\frac{y^2}{x^2}\right)^{-3/2} \quad (21)$$

$$\lim_{x\to 0^+} F(x,y) = -\frac{\pi x^2}{2x_0^2}\left(1+\frac{y^2}{x_0^2}\right)^{-3/2}. \quad (22)$$

The kinetic energy density due to the vortex thus falls off as x^{-5} at large x for fixed y, and as xy^{-6} for large y and fixed x, which allows us to conclude that vortices near Thomas-Fermi surfaces are essentially localized structures that do not extend into the condensate beyond a depth of order x_0.

Inner asymptotics

For matching with the inner zone solution, which will determine the vortex velocity, we need the behaviour of $Q_{1/2}(z)$ as $z \to 1^+$. This can be obtained analytically. Using dimensionless polar co-ordinates (r, ϕ) centred on the vortex,

$$x = x_0 + \xi r \cos \phi \ , \ y = \xi r \sin \phi, \qquad (23)$$

and taking $\varepsilon = \xi/x_0$, we can express z as we approach the vortex as

$$\begin{aligned} z\left(1 + \varepsilon r \cos\phi, \varepsilon r \sin\phi\right) &= 1 + \frac{\varepsilon^2}{2}\frac{r^2}{1 + \varepsilon r \cos\phi} \\ &\equiv 1 + \frac{\varepsilon^2 \Delta^2}{2}. \end{aligned} \qquad (24)$$

We can then use this expansion of z in the integral representation [32] of the Legendre function

$$Q_\nu(z) = \int_0^\infty \frac{ds}{\left(z + \sqrt{z^2 - 1}\cosh s\right)^{\nu+1}} \qquad (25)$$

to obtain

$$Q_{\frac{1}{2}}(1 + \varepsilon^2 \Delta^2/2) = \mathcal{O}(\varepsilon^2) + \int_0^\infty \frac{ds}{(1 + \varepsilon\Delta\cosh s)^{3/2}} \quad (26)$$

Then we can note that

$$\begin{aligned} &\int_0^\infty \frac{ds}{(1 + \varepsilon\Delta\cosh s)^{3/2}} \\ &= \int_0^{-\frac{\ln\varepsilon\Delta}{2}} \frac{ds}{(1 + \varepsilon\Delta\cosh s)^{3/2}} \\ &+ \int_{-\frac{\ln\varepsilon\Delta}{2}}^\infty \frac{ds}{\left(1 + e^{s + \ln\frac{\varepsilon\Delta}{2}} + e^{-s+\ln\frac{\varepsilon\Delta}{2}}\right)^{3/2}} \end{aligned} \quad (27)$$

and evaluate each of the two integrals using different expansions in ε (thus evaluating (25) using a kind of minature boundary layer theory of its own!). That is, in the first integral of the RHS of (27) we simply expand in $\varepsilon\Delta$ and integrate term by term; and in the second we use

$$\begin{aligned} &\int_{-\frac{\ln\varepsilon\Delta}{2}}^\infty \frac{ds}{\left(1 + e^{s + \ln\frac{\varepsilon\Delta}{2}} + e^{-s + \ln\frac{\varepsilon\Delta}{2}}\right)} \\ &\doteq \int_{\ln\frac{\sqrt{\varepsilon\Delta}}{2}}^\infty \frac{ds}{(1 + e^s)^{3/2}} \\ &= \left[\frac{2}{\sqrt{1 + e^s}} + \ln\frac{\sqrt{1 + e^s} - 1}{\sqrt{1 + e^s} + 1}\right]_{\ln\frac{\sqrt{\varepsilon\Delta}}{2}}^\infty \end{aligned} \quad (28)$$

where in the second line, as hereafter, $\doteq$ means dropping a term of order ε^2.

Combining both results we obtain

$$\begin{aligned} Q_{\frac{1}{2}}(1 + \varepsilon^2\Delta^2/2) &\doteq -\left(2 + \ln\frac{\varepsilon\Delta}{8}\right) \\ &\doteq -2 - \ln\frac{\varepsilon r}{8} + \frac{\varepsilon r \cos\phi}{2}, \end{aligned} \quad (29)$$

and hence

$$\begin{aligned} &F(x_0 + \xi r \cos\phi, \xi r \sin\phi) \\ &= \left(1 + \frac{\varepsilon r \cos\phi}{2}\right)\ln\frac{\varepsilon r}{8} \\ &+ 2 + \frac{\varepsilon}{2}r\cos\phi + \mathcal{O}(\varepsilon^2). \end{aligned} \quad (30)$$

Translating from the dual potential F back to $\vec{A}$ according to (5), to first order in ε we therefore have

$$\lim_{\vec{x}\to\vec{x}_0} A_r \doteq \frac{1}{\xi}\left[\frac{1}{r} + \frac{\varepsilon\cos\phi}{2}\ln\frac{\varepsilon r}{8}\right] \qquad (31)$$

$$\lim_{\vec{x}\to\vec{x}_0} A_\phi \doteq -\frac{1}{\xi}\left[\frac{\varepsilon}{2}\left(1 + \ln\frac{\varepsilon r}{8}\right)\right]\sin\phi. \qquad (32)$$

Applying (4) we then see

$$\begin{aligned} \lim_{\vec{x}\to\vec{x}_0}\partial_\phi\theta &\doteq 1 + \frac{\varepsilon r\cos\phi}{2}\ln\frac{\varepsilon r}{8} \\ \lim_{\vec{x}\to\vec{x}_0}\partial_r\theta &\doteq \frac{\varepsilon}{2}\left(1 + \ln\frac{\varepsilon r}{8}\right)\sin\phi \end{aligned}$$

from which we can obviously extract the condensate phase field near the vortex,

$$\theta(r, \phi) = \phi + \varepsilon r\frac{\sin\phi}{2}\ln\frac{\varepsilon r}{8} + \mathcal{O}(\varepsilon^2). \qquad (33)$$

Note that there is no trace of Euler's constant in this equation.

INNER SOLUTION: GROSS-PITAEVSKII

The perturbative problem

In the inner zone we work entirely in terms of the dimensionless polar co-ordinates centered on the vortex: $x = x_0 + \xi r\cos\phi, \ y = \xi r\sin\phi$. (Since the actual condensate density at the vortex center vanishes, it is perhaps worth clarifying that for the local healing length 'at' the vortex $\xi = (g\rho_0)^{-1/2}$ we use for ρ_0 the background Thomas-Fermi density extrapolated to the vortex location.) To obtain our inner region solution, we expand the potential to first order about the vortex position: $V(\vec{x}) = \lambda x = \lambda x_0\left[1 + \varepsilon r\cos\phi + \mathcal{O}(\varepsilon^2)\right]$. It will then be convenient to rescale the wave function, defining

$$\Psi = e^{-i\mu t/\hbar}\sqrt{\rho_0}\psi\left(\vec{x} - \varepsilon\vec{\beta}ct\right) \qquad (34)$$

where $c = \hbar/(M\xi)$ is the local speed of sound at the vortex, and $\vec{v}_{vtx} = \varepsilon\vec{\beta}c$ is the vortex velocity, as yet unknown. (We have reduced the number of symbols to be defined by anticipating the fact that the vortex velocity will be order ε.) Then writing $\psi = \psi_0 + \varepsilon\psi_1 + ...$, we find

$$-\frac{1}{2}\left[\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\phi^2\right]\psi_0 + \left[|\psi_0|^2 - 1\right]\psi_0 = 0 \quad (35)$$

as our zeroth order equation. The only solution which corresponds to a singly quantized vortex at $r = 0$ is $\psi_0 = f(r)e^{i\phi}$ where $f(r)$ can be taken as real, and satisfies

$$\frac{1}{2}\left[\partial_r^2 + \frac{1}{r}\partial_r\right]f = \frac{f}{2r^2} + \left[f^2 - 1\right]f \quad (36)$$

and $f(r \to \infty) = 1$. Note that a vortex velocity of order ε^0 would require a $\vec{v}_{vtx} \cdot \vec{\nabla}\psi_0$ term on the RHS of (35), and the only well-behaved vortex solution with such a term present is $e^{iM\vec{v}\cdot\vec{x}/\hbar}\psi_0$; but this would imply a phase gradient of order ε^0 as one approaches the outer zone ($r \to R/\xi$), and this would be inconsistent with our outer solution (33).

At first order, though, we find

$$-\frac{1}{2}\left[\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\phi^2\right]\psi_1 + \left[2f^2 - 1\right]\psi_1 + \psi_0^2\psi_1^*$$
$$= r\cos\phi\,\psi_0 - i\left(\beta_x\cos\phi + \beta_y\sin\phi\right)\partial_r\psi_0$$
$$+ \left(\beta_y\cos\phi - \beta_x\sin\phi\right)\frac{\psi_0}{r}. \quad (37)$$

It is not easy to solve this equation for ψ_1; but we are actually only interested in two pieces of information. We need to know the asymptotic behaviour of ψ_1 many healing lengths away from the vortex center, so that we can smoothly match the inner solution to the outer. And we need to determine $\vec{\beta}$, which will be fixed by the requirement that ψ_1 does not blow up as r increases. Following Rubinstein and Pismen [7], we will be able to obtain this information without solving (37) explicitly.

At large r we can write $\psi_1 \to e^{i\phi}[S + iT]$ and find $S \to \frac{r}{2}\cos\phi$. Then since $f \to 1 - (2r)^{-2}$ at large r, the leading terms in T are driven by the $-ir^{-2}\partial_\phi S$ crossterm, giving the asymptotic equation $\nabla^2 T = r^{-1}\sin\phi$, with the solution

$$T \to r\left(\alpha_x\cos\phi + \alpha_y\sin\phi\right) + \frac{r}{2}\ln r\sin\phi \quad (38)$$

for coefficients $\vec{\alpha}$ that will be fixed by matching with the outer zone. (We drop a constant which can obviously be absorbed in ψ_0, and which is of no consequence anyway.) So we have

$$\lim_{r\to\infty}\Psi = e^{i\phi}\sqrt{\rho_0}[1 + \varepsilon\frac{r}{2}(\cos\phi + i\ln r\sin\phi)$$
$$+ i\varepsilon\vec{\alpha}\cdot\vec{r}]$$
$$= \sqrt{\rho}\exp i[\phi + (\varepsilon r/2)(\ln r + 2\alpha_y)\sin\phi$$
$$+ \varepsilon r\alpha_x\cos\phi]$$
$$= \sqrt{\rho}\exp i\left(\phi + \varepsilon\frac{r}{2}\ln\frac{\varepsilon r}{8}\sin\phi\right) \quad (39)$$

dropping terms of $\mathcal{O}(\varepsilon^2)$. In the last line we have imposed matching with the outer solution (33) to fix

$$\alpha_x = 0 \quad, \quad \alpha_y = \frac{1}{2}\ln\frac{\varepsilon}{8}. \quad (40)$$

To constrain $\vec{\beta}$, note that differentiating (35) with respect to x or y shows that $\partial_x\psi_0$ and $\partial_y\psi_0$ are two independent solutions to the homogeneous equation for ψ_1. Writing $\mathcal{E}$ as an abbreviation for the left-hand side of (37), and $\mathcal{J}$ as an abbreviation for the right-hand side, we integrate both sides of (37) with $\partial_x\psi_0^*$ out to the large dimensionless radius R/ξ. Integrating by parts and using our results for ψ_1 at large radius reveals

$$\text{Re}\left[\int_0^{\frac{R}{\xi}}rdr\oint d\phi\left[e^{-i\phi}\left(f'\cos\phi + i\frac{f}{r}\sin\phi\right)\mathcal{E}\right]\right]$$
$$= \frac{1}{2}\text{Im}\oint d\phi\left[e^{-i\phi}\sin\phi\left(\partial_r\psi_1 + r^{-1}\psi_1\right)\right]_{r=R/\xi}$$
$$= \frac{\pi}{2}\left[\ln\frac{\varepsilon R}{8\xi} + \frac{1}{2}\right] = \frac{\pi}{2}\left[\ln\frac{R}{\xi} + \frac{1}{2} + 2\alpha_y\right]$$
$$= \text{Re}\left[\int_0^{\frac{R}{\xi}}rdr\oint d\phi\left[e^{-i\phi}\left(f'\cos\phi + i\frac{f}{r}\sin\phi\right)\mathcal{J}\right]\right]$$
$$= \pi\int_0^{\frac{R}{\xi}}dr\,ff'\left[r^2 + 2\beta_y\right]$$
$$= \pi\left[\beta_y + \frac{R^2}{2\xi^2}(1 - \frac{\xi^2}{2R^2}) - \int_0^{\frac{R}{\xi}}dr\,rf^2\right]. \quad (41)$$

This allows us to obtain

$$\beta_y - \alpha_y = \lim_{R/\xi\to\infty}\left[\frac{1 + \ln(R/\xi)}{2} - \int_0^{\frac{R}{\xi}}dr\,r(1 - f^2)\right]$$
$$= \frac{1}{2}\left[1 + \ln\frac{R}{\xi} + \int_0^{\frac{R}{\xi}}dr\left[r\frac{f''}{f} + \frac{f'}{f} - \frac{1}{r}\right]\right]$$
$$\doteq \frac{1}{2}\left[1 - \ln f'(0) + \int_0^\infty dr\,r\frac{f''}{f}\right]$$
$$\doteq -0.114. \quad (42)$$

In the second line we have used (36) to re-write $(1 - f^2)$ in the integrand, and in the last line the numerical evaluation comes from a numerical solution for $f(r)$. Similarly, using the $\partial_y\psi_0 = e^{i\phi}\left[f'\sin\phi + ir^{-1}f\cos\phi\right]$ solution, we find

$$\beta_x = 0. \quad (43)$$

(The numerical result in (42) was obtained with *Mathematica* [33], using two different methods of numerical solution for $f(r)$ (shooting and relaxation), whose results agreed with each other. Quite stringent settings of the options for starting step sizes, etc., were required to obtain this agreement, and in both methods the singularities at $r = 0$ had to be regulated, for example by

replacing $r \to \sqrt{r^2 + 10^{-20}}$ in the singular co-efficients in (36). Our result does not quite agree with the evaluation reported in [7]: their value of 0.405 for their quantity $\ln a_1$ corresponds to a value of -0.126 for our quantity $\beta_y - \alpha_y$.)

Combining (40) and (42), we obtain the total vortex velocity. It is in the negative y direction: parallel to the TF surface, and in the direction of the fluid flow between the vortex and the surface. Extracting explicitly the x_0 dependence hidden in $\varepsilon = \sqrt{2\delta^3/x_0^3}$, we can express the magnitude v_{vtx} of the vortex velocity purely in terms of x_0 and surface parameters:

$$
\begin{aligned}
v_{vtx} &= \varepsilon c\,|\beta| = \frac{\xi}{x_0} \frac{\hbar}{M\xi}|\beta| \\
&= \frac{\hbar}{Mx_0}\left[\frac{1}{4}\ln\left(\frac{32x_0^3}{\delta^3}\right) + 0.114\right] \\
&= v_c \frac{\delta}{x_0}\left[\frac{3}{4}\ln\left(\frac{x_0}{\delta}\right) + 0.980\right]
\end{aligned} \tag{44}
$$

where

$$
v_c = \frac{\hbar}{M\delta} \tag{45}
$$

is the surface characteristic (and critical [24]) velocity. Eqn. (44) is the first main result of this paper; it is illustrated graphically by Figure 3. It will be accurate for vortices in quasi-two-dimensional condensates as long as the vortex distance from the TF surface x_0 is much larger than the surface depth δ but much smaller than the TF radius. For quasi-2D condensates of size comparable to current three-dimensional condensates, this will be a significant regime of validity. Of course, even apart from its realism, (44) remains instructive as an accurate result for an idealized problem.

Free energy and vortex penetration

If dissipation occurs in a frame moving with respect to the condensate with velocity v_{dis}, then the vortex will drift towards or away from the surface, in order to minimize the free energy

$$
G = \frac{\pi\hbar^2}{2Mg\delta^2}\left(E - p\frac{v_{dis}}{v_c}\right), \tag{46}
$$

where the prefactor is a surface energy scale, so that the dimensionless energy E and y component of momentum p are given by

$$
E = \frac{x_0}{\pi\rho_0\delta}\int d^2x \left[\frac{1}{2}\left|\vec{\nabla}\Psi\right|^2 + \frac{g}{2}\left(|\Psi|^2 - \frac{Fx}{g}\right)^2\right] \tag{47}
$$

$$
p = \frac{x_0}{\pi\rho_0\delta^2}\,\mathrm{Im}\int dxdy\;\Psi^*\partial_y\Psi. \tag{48}
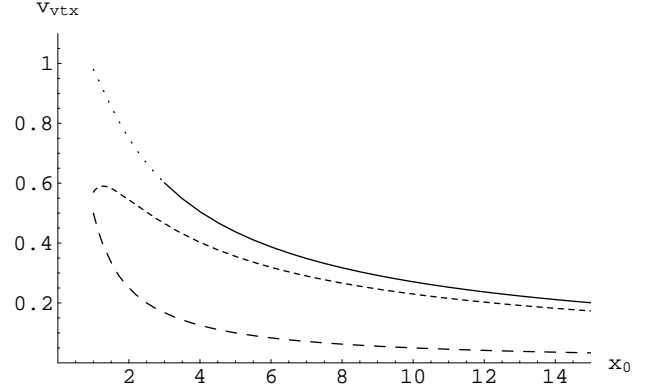$$

FIG. 3: Velocity of a vortex parallel to a Thomas-Fermi surface in a quasi-two-dimensional condensate (upper curve). The vertical axis is velocity in units of the surface mode critical velocity $\hbar/(M\delta)$; the horizontal axis is distance of the vortex center from the TF surface, in units of the surface depth δ. As a leading order result in δ/x_0, the curve is not meaningful to the left $x_0 = \delta$. The dashed curves, presented for comparison, are the Rubinstein-Pismen result (short dashes), and the velocity $\hbar/(2Mx_0)$ of a vortex the same distance from a hard wall in a condensate of constant density (long dashes). The sum of the two dashed curves is quite a good approximation to the solid curve.

In this expression for E we have subtracted off the Thomas-Fermi free energy of the vortex-free condensate. We will now evaluate this free energy to leading order in ε using our results from above. As above, we will compute the inner and outer zone contributions separately, and add them. In this sum, all dependences on the inner zone size $R \sim \varepsilon^{-1/2}\xi$ necessarily cancel out.

Computing E

Since we know that ν_{vtx} is of order ε, it is not hard to see that the order ε terms in the inner component E_{in} of E vanish, and up to corrections of order ε^2 we have

$$
\begin{aligned}
E_{in} &= \frac{x_0}{\delta}\int_0^{\frac{R}{\xi}} dr\, r\left[f'^2 + \frac{f^2}{r^2} + \left(f^2 - 1\right)^2\right] \\
&= \frac{x_0}{\delta}\int_0^{\frac{R}{\xi}} dr\, \partial_r\left[rff' + r^2f^2 - \frac{r^2f^4 + f^2 - r^2f'^2}{2}\right] \\
&\quad + \frac{x_0}{\delta}\int_0^{\frac{R}{\xi}} dr\, r\left(1 - 2f^2\right) \\
&= \frac{x_0}{\delta}\left[\ln\frac{R}{\xi} - 2\left(\beta_y - a_y\right) + \frac{1}{2}\right]
\end{aligned} \tag{49}
$$

where in the last line we also drop terms of order $(\xi/R)^2$, which are not only of order ε, but will also be cancelled by terms from the outer component E_{out}.

In the outer zone we must integrate the energy density over the entire half plane, except for the inner zone circle

of radius $R \sim \varepsilon^{1/2}x_0$. Denoting integration over this region with the subscript $\mathcal{A}$ we have, to leading order in ε

$$
\begin{aligned}
E_{out} &= \frac{x_0}{\pi\delta}\int_{\mathcal{A}} dxdy \, \frac{\rho}{\rho_0}\left|\vec{\nabla}\theta\right|^2 = \frac{x_0}{\delta}\int_{\mathcal{A}} dxdy \, \frac{x_0}{x}\left|\vec{\nabla}F\right|^2 \\
&= \frac{x_0^2}{\pi\delta}\int_{\mathcal{A}} dxdy \, \vec{\nabla}\cdot\left(\frac{F}{x}\vec{\nabla}F\right) = \frac{x_0^2}{\pi\delta}\oint_{\partial\mathcal{A}} d\vec{S}\cdot\frac{F}{x}\vec{\nabla}F \\
&= -\frac{x_0}{\pi\delta}\oint_{r=R/\xi} d\phi \, F\partial_r F \\
&= -\frac{x_0}{\delta}\left(\ln\frac{\varepsilon R}{8\xi}+2\right).
\end{aligned}
\tag{50}
$$

To obtain the second line from the first we use (11), where the RHS vanishes everywhere in $\mathcal{A}$. Within the second line we use the divergence theorem, where the surface terms at $x=0$ and at infinity all vanish, so that the only contribution is on the boundary with the inner zone at the circle of radius R about x_0. So putting E_{in} and E_{out} together we have

$$
\begin{aligned}
E &= -\frac{x_0}{\delta}\left(\frac{3}{2}+2\beta_y\right) \\
&= 2\frac{x_0}{\delta}\left[\frac{3}{4}\left(\ln\frac{x_0}{\delta}-1\right)+0.980\right].
\end{aligned}
\tag{51}
$$

Writing E as we have in the last line emphasizes that dE/dx_0 is proportional to v_{vtx}, indicating that x_0 and v_{vtx} are canonically conjugate, as one expects for a vortex.

Computing p

To compute p it is very helpful to note that the dual potential F can be extended into the inner zone, by using the continuity equation and the fact that the density is constant in the frame co-moving with the vortex to obtain

$$
\begin{aligned}
0 &= (M/\hbar)\left[v_{vtx}\partial_y\rho-\dot{\rho}\right] \\
&= \vec{\nabla}\cdot\left[\rho\left(\vec{\nabla}\theta+\frac{M}{\hbar}\hat{y}v_{vtx}\right)\right]
\end{aligned}
\tag{52}
$$

which allows us to define

$$
\rho\partial_y\theta = \rho_0\partial_x F+\frac{M}{\hbar}v_{vtx}\left(\rho-\frac{x}{x_0}\rho_0\right)
\tag{53}
$$
$$
\rho\partial_x\theta = -\rho_0\partial_y F .
\tag{54}
$$

Taking this definition of F into the outer zone, it clearly agrees with our previous one up to post-hydrodynamic corrections. (Alternatively one could repeat the outer zone analysis using this slightly different definition of F, and confirm that no differences arose up to order ε^2.) Furthermore one can show from the Gross-Pitaevskii

equation that this extended F is perfectly regular as $r\to 0$, where it behaves as $\int dr \, r^{-1}f^2$. We can therefore write

$$
\begin{aligned}
p &= \frac{x_0}{\pi\delta^2}\int dxdy \, \frac{\rho}{\rho_0}\partial_y\theta \\
&= \frac{x_0}{\pi\delta}\frac{v_{vtx}}{v_c}\int dxdy\left(\frac{\rho}{\rho_0}-\frac{x}{x_0}\right) \\
&\quad -\frac{x_0}{\pi\delta^2}\int_{-\infty}^{\infty} dy\left[F(0,y)-\lim_{x\to\infty}[F(x,y)]\right].
\end{aligned}
\tag{55}
$$

The leading contribution to p is the last line of (55). Its evaluation is very simply obtained from (21):

$$
\begin{aligned}
p &\doteq \frac{x_0}{\pi\delta^2}\lim_{x\to\infty}\int_{-\infty}^{\infty} dy \, F(x,y) \\
&= -\frac{x_0^2}{2\delta^2}\lim_{x\to\infty}\int_{-\infty}^{\infty}\frac{dy}{x}\left(1+\frac{y^2}{x^2}\right)^{-3/2} \\
&= -\frac{x_0^2}{2\delta^2}\int_{-\infty}^{\infty}\frac{dy}{(1+y^2)^{3/2}} = -\frac{x_0^2}{\delta^2}.
\end{aligned}
\tag{56}
$$

This leaves out the density deficit integral in the second line of (55), which can be shown to be of order δ/x_0, and hence smaller than the leading term by a factor of ε^2.

Free energy curve

We therefore have, up to corrections of order ε^2,

$$
G = \frac{\pi\hbar^2}{2Mg\delta^2}\left[\frac{x_0}{\delta}\left(0.46+\frac{3}{2}\ln\frac{x_0}{\delta}\right)+\frac{v_{dis}}{v_c}\left(\frac{x_0}{\delta}\right)^2\right]
\tag{57}
$$

which is plotted in Figure 4. At least within the hydrodynamic approximation, the energetic barrier to vortex penetration has clearly disappeared for $|v_{dis}|>v_c$. This is the second main result of this paper. Since we are considering a vortex with counter-clockwise circulation, located along the positive x axis away from a TF surface on the y axis, it is to be expected that a negative v_{dis} makes vortex penetration become energetically favourable.

Application to rotating harmonic traps

For a harmonically trapped condensate of spatial size R_{TF}, the nonlinearity of the trapping potential near the TF surface, and the curvature of the TF surface, can both be neglected in the limit where $x_0/R_{TF}\to 0$. Hence our results can be applied to harmonically trapped condensates if we regard them as leading order approximations in both δ/x_0 and x_0/R_{TF}. If we consider a rotational symmetric trap, we take R_{TF} to be the Thomas-Fermi radius in the plane of symmetry, and then assume either a quasi-two-dimensional 'pancake' trap, or a quasi-cylindrical 'extreme cigar' trap in which vortex lines are
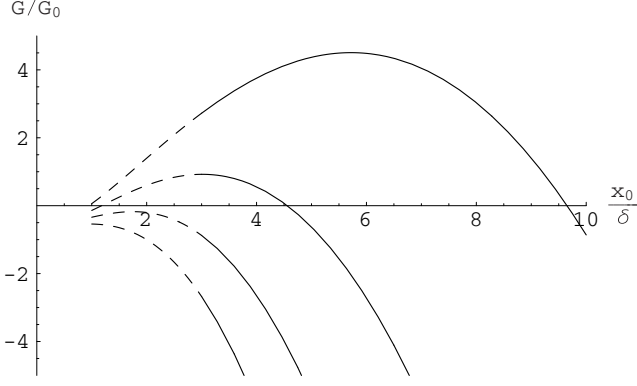
FIG. 4: Free energy $G/G_0 = E - pv_{dis}/v_c$, where $G_0 = \pi\hbar^2/(2Mg\delta^2)$, $\delta = (2M\lambda/\hbar^2)^{-1/3}$, and $v_c = \hbar/(M\delta)$. Horizontal axis is x_0/δ, distance of vortex center from Thomas-Fermi surface in units of surface depth δ. From uppermost to lowermost, the curves are for $v_{dis}/v_c = -0.4, -0.6, -0.8, -1$. The plots are shown dashed for $x_0/\delta < 3$, and stop at $x_0/\delta = 1$, to reflect that the curves are leading order results in δ/x_0.

parallel to the long axis z. In the latter case the z length of the condensate will only appear as an overall factor in the free energy, and so effectively these two cases are exactly similar. We can translate our parameters into those commonly used for harmonic trapped condensates by noting that the potential gradient at the TF surface is just

$$\lambda = M\omega^2 R_{TF} \tag{58}$$

where ω is the radial trap frequency. This yields

$$\delta = \left(\frac{\hbar^2}{2M^2\omega^2 R_{TF}}\right)^{1/3} \tag{59}$$

and

$$\varepsilon = \frac{\hbar}{M\omega\sqrt{x_0^3 R_{TF}}} = \frac{a_0^2}{x_0^{3/2} R_{TF}^{1/2}} \tag{60}$$

for the trap size $a_0 = (M\omega/\hbar)^{-1/2}$. If we estimate the size of ε in an effectively 2D condensate by supposing dimensions typical of current large three-dimensional condensates, taking $a_0 \sim \mu$m and $R_{TF} \sim 25\mu$m gives $\varepsilon \lesssim 0.2$ for $x_0 \gtrsim a_0$. So we see that the calculation should be accurate for vortices a few microns inside the TF surface. (It should be emphasized once again that three dimensional effects such as bending of vortex lines are known to be present in current experiments, but are entirely absent in our model. It seems reasonable to hope that vortex bending will be slight and have small impact in highly prolate or highly oblate traps, which should approach either our 2D or cylindrical limits; but it is difficult to extend the present analysis far enough to support this hope

with calculations. The prospect that an initially straight vortex line might bend more and more as it moved seems difficult to rule out.)

We can also obviously interpret $v_{dis} = \Omega R_{TF}$ where Ω is the frequency at which a perturbation to the potential is rotated in order to stir the condensate. With this translation of v_{dis}, we can compare Figure 4 to the right-hand edge of Figure 5 from Reference [13], and compare (57) to Eqn. (49) of [13] ('SF49') in the limit where $\zeta_0 \to 1 - 2x_0/R$ in that equation. An essentially similar equation is presented in [14] as their Eqn. (9). (Both these works examine three-dimensionally harmonic traps, in which, however, vortex curvature is neglected, so that the length of the vortex line is proportional to $x_0^{1/2}$. Hence in their cases energy and momentum scale as $x_0^{3/2}$ and $x_0^{5/2}$, respectively, rather than x_0 and x_0^2 as in ours.) Equation SF49 was derived by assuming that, for a rotationally symmetric trap, the phase field of the vortex would be well approximated throughout the condensate by a simple $e^{i\phi}$ with ϕ the usual polar co-ordinate centered on the vortex. Unless the vortex is very close to the center of the trap, this *ansatz* violates continuity significantly over most of the condensate. And we know from our results above that it significantly exaggerates the degree to which the velocity field of a vortex centered near the TF surface extends into the bulk of the condensate. Hence according to SF49, the rotation frequency Ω_ν [1] at which the free energy becomes negative for all $x_0 > \delta$ is an overestimate (*i.e.* is greater than v_c/R_{TF}) by the significant factor $(5/4)\ln(R_{TF}/\xi_0)$. So while discussions of vortex free energies based on the simple $e^{i\phi}$ ansatz will typically be qualitatively sound, they can easily be inaccurate by factors of two or more, and cannot be used to obtain precise predictions for critical velocities or stirring frequencies.

DISCUSSION

Magnus effect and infrared dressing

The vortex velocity component $\beta_y - \alpha_y$ is perhaps the least trivial aspect of the vortex motion, inasmuch as it describes a component of the motion that is determined by the inner zone analysis alone, independent of the background fluid velocity α_y in its immediate neighbourhood. (The latter is nontrivial to determine, from the outer zone hydrodynamics, but trivial in the way it moves the vortex). As first pointed out by Rubinstein and Pismen [7], this intrinsic motion is along contours of constant Thomas-Fermi density. We can understand it qualitatively by noting that the vortex is a bubble-like density defect, and so experiences a buoancy force in the opposite direction to the trap force. Due to the Magnus effect which dominates vortex motion in superfluids and

normal fluids alike, this force produces not an acceleration, but a velocity at right angles:

$$\vec{v}_{Mag} = -\frac{\vec{F} \times \hat{z}}{M\kappa\rho} = -\frac{\vec{F} \times \hat{z}}{2\pi\hbar\rho}, \qquad (61)$$

since in our case the vortex circulation κ is $2\pi\hbar/M$. (If we had taken a vortex swirling in the opposite direction, we would indeed have found β_y to have the opposite sign.)

The precise magnitude of the buoancy force, and hence of the vortex velocity, is nontrivial, because the naive buoancy force

$$\begin{aligned} \vec{F} &= \vec{\nabla}V \int d^2x \left(\rho_0 - |\Psi|^2\right) \\ &= -\hat{x}\varepsilon c \left(2\pi\rho\hbar\right) \int dr\, r(1 - f^2) \end{aligned} \qquad (62)$$

is logarithmically divergent. What [7] showed first, and our own inner zone analysis has reproduced, is that the buoancy force is renormalized. The longer range effects of the potential gradient produce a logarithmic distortion of the vortex flow pattern, and since the classic Magnus effect is for a strictly cylindrical flow, this distortion produces the counterterms $(1 + \ln R/\xi)/2$ in the first line of (42).

On the other hand, we can see from (39) that the flow distortion in ψ_1 implies a velocity field component $\propto \hat{y}\ln r$, which is essentially indistinguishable in any range of r from a background velocity perpendicular to the potential gradient. So the distinction between the intrinsic $(\beta_y - \alpha_y)$ and ambient α_y components of vortex motion is somewhat artificial, and it is more natural to consider the whole flow pattern as a 'dressed vortex'.

Previous analytic studies of vortices in BECs have often neglected this dressing effect, by assuming that the phase field $e^{i\theta(\vec{x})}$ of a vortex always remains $e^{i\phi}$ in polar co-ordinates centred on the vortex. Close to the core of a vortex this is indeed a good approximation, as long as 'close' means that the background condensate density has not varied appreciably. If further away from the core the background density varies (and is not rotationally symmetric about the vortex), then thia simple *ansatz* for the phase field obviously fails to satisfy the continuity condition. Hence at distances from the vortex core that are on the scale of the trapping potential's variation, the phase field will depart significantly from $e^{i\phi}$. Furthermore, since the corrections to $e^{i\phi}$ are caused by density variations on the potential scale, their own spatial scale will be of this same order. And this means that the corrections will extend into the vortex core region, in the form a 'local ambient' flow. This flow is constant on the healing length scale, but it exists just because the vortex is present in the inhomogeneous sample, and in this sense can be considered part of the vortex: it is a component of the vortex's infrared dressing.

Against image vortices

The simplest example of this infrared dressing phenomenon is already well known, as it occurs in the special case of inhomogeneity that is a 'hard wall' boundary on an otherwise homogeneous sample. Here the nontrivial requirement of continuity far from the vortex is simply the constraint that there be no flow through the hard wall surface. In this special case the hydrodynamic problem to be solved for the vortex phase field is simply the Laplace equation, with Dirichlet boundary conditions on the surface. For many shapes of surface, the method of images is a technical trick that produces the required solution. This convenient technique has perhaps had the unfortunate side-effect of obscuring the general phenomenon of vortex infrared dressing, by giving rise to an impression that 'image vortices' are the only significant effect of inhomogeneity. In fact the method of images is restricted to conveniently symmetrical equations and boundary conditions, and it requires an exact solution which is known in the Laplace case, but not in general. It is of no help in the case of a Thomas-Fermi surface, because the single vortex solution automatically satisfies the correct boundary condition. And it is not even applicable for a hard wall surface, unless the density profile within the wall has a fortunate form. (For example, the solution obtained in this paper will allow image solutions for a hard wall surface perpendicular to the gradient of a linear potential, but not for walls at other angles.)

Thinking about image vortices will typically allow a qualitative understanding of how vortices will move near a surface; but this will not be quantitatively reliable. For those strongly attached to the image vortex picture, the singularity of $Q_\nu(z)$ at $z = -1$, corresponding to $x = -x_0$, may reveal an image vortex even in the case solved above. But since image vortices are actually just a mathematical trick, for representing the physical effect of a surface, it would arguably be just as well not to rely on them as a conceptual tool beyond their regime of real applicability. Instead of picturing repulsive or attractive image vortices, it is not so hard just to think about the accelerated flow through the 'channel' between a vortex and a surface, and so obtain a qualitative understanding that is equally convenient and more genuine.

On the other hand, using the Laplace image vortex velocity $\hbar/(2Mx_0)$ to estimate the size of surface effects on vortices is obviously better than nothing, if the full hydrodynamic problem is intractable (as it may well be). Combining this crude image vortex theory with the local theory of Rubinstein and Pismen [7] does seem to work surprisingly well for the particular case analysed in this paper; but there are no grounds for expecting it to be generally accurate. Using such uncontrolled approximations to guide experimental design might be reasonable, given the many unknown factors present in the

early stages of an experiment. But even fairly substantial disagreements between experimental measurements, and theoretical predictions based on such zeroth-order estimates, would be no evidence of novel phenomena.

Estimating the critical velocity for vortex 'nucleation'

Since the vortex flow field in a homogeneous sample is already long ranged, it is possible that the infrared dressing which occurs in an inhomogeneous background may distort the long ranged velocity field by actually reducing its extent. We have seen in this paper that for a vortex near a Thomas-Fermi surface this sort of screening effect does indeed occur, so that the vortex is a well localized structure. As we discussed briefly at the end of the last Section, this effect can be a significant correction on estimates of the velocity of stirring needed to drive vortices into a condensate.

Since surface Bogoliubov modes can be considered as the motion of vortices in the ultra-dilute tail of the condensate density profile extending outside the TF surface, the present paper can be regarded as a complement to [24], showing how vortices may behave soon after they have 'nucleated' (that is, entered the condensate cloud). As we have mentioned, our hydrodynamic and boundary layer approximations are only valid if the depth of the vortex inside the TF surface is much greater than the local surface depth, and also much smaller than the sample size. (Hence the applicability of the plane linear model for vortices inside the condensate is somewhat more restricted than for the Bogoliubov surface mode spectrum in [24], where it is only required that the surface length be much less than the sample size.) Because of its inapplicability at small x_0, Figure 4 is not really adequate to derive a precise result for the critical v_{dis} above which vortices will tend to enter the condensate spontaneously; but one can deduce from it with good confidence that this critical value must be above $v_c/2$. The lowest curve of Figure 4 certainly shows that, above the surface mode critical velocity v_c, no barrier to vortex penetration will remain within the hydrodynamic region. This implies that the limitation on vortex penetration is in the perturbative region, where the surface mode analysis of [24] fixes the critical velocity at $v_c = \hbar/(M\delta)$. And this means that vortices enter a condensate through an ordinary instability, and not by a quantum or thermal barrier-crossing process, which is usually what is meant by the term 'nucleation'. But two warnings must be attached to this conclusion.

The first is that neither the surface mode analysis of [24] nor the boundary layer theory of the present paper are valid for vortices within a few δ of the TF surface, and so one might in principle worry that a narrow energetic barrier might still exist, within this region, at velocities above v_c. Resolving this concern analytically would be very difficult, but it can be dismissed with physical arguments. The TF surface is not a physical surface, not a skin or a wall; it is a place where the condensate density almost vanishes. In the neighbourhood of the TF surface one can expect to see a transition between inner and outer regimes, but there is simply not enough of anything there for this interface to constitute a third regime of its own. Furthermore, vortices whose centers are within a surface depth of the surface have core sizes on the order of δ as well, so that a vortex centered on the TF surface already extends into the hydrodynamic regime on one side, and the perturbative regime on the other. This makes it very implausible that a barrier arises at the TF surface, when the free energy is monotonic in the same direction on both sides of it. And, finally, experience with numerical integration of dissipative 2D Gross-Pitaevskii equations has always shown that once vortices begin sinking towards the TF surface, they pass through it and enter the condensate without any difficulty. Hence we conclude that at velocities above v_c no barrier exists for vortices, at least in condensates large enough for the linear density profile to be an accurate local model.

The second warning is more important, which is that three dimensional effects may perhaps hold vortex loops near the TF surface, simply because for a vortex line to sink more deeply into a condensate it must typically grow in length, adding the vortex 'string tension' to the free energy. If the vortex line bends and loops, this effect might be much greater than can be accounted for by the factors of $x_0^{1/2}$ allowed in recent 3D calculations assuming straight vortex lines. Just how strong such an effect might be is far beyond the scope of this paper; but there have been some indications in experiments that stirring a condensate at just above the critical frequency generates twisting vortex loops that remain near the edges of the cloud [34].

Summary and outlook

This paper has presented an exactly solvable hydrodynamic problem, namely the case of a point vortex in a plane linear background density profile, which can supplement the hard wall as an example on which to base understanding of general cases of vortices interacting with surfaces. This particular solution is also directly relevant to the real problem of a vortex near the Thomas-Fermi surface of a large condensate. It will even be an accurate approximation in a realizable regime, because the velocity field in this case actually falls off much more quickly than in the homogeneous case, so that the longer-range effects of nonlinear potentials and curved surfaces may be neglected more generally than one might initially have expected.

It is unfortunate, however, that the two conditions of a density gradient and a surface are combined in this solvable case, because it therefore does not afford us much intuition about how their effects may differ. It should be clear from the preceding discussion that the effects found in this case are not purely surface effects: if at some point the linear trapping potential levelled off and then diminished, the TF surface would be replaced by a mere local minimum in background density, but the density gradient alone would still have effects on vortex dynamics. Vortices with density inhomogeneity far from any surfaces should be analytically tractable in some simple models, such as slightly varying periodic potentials. A few aptly chosen numerical examples might be almost as instructive, and require less effort.

Acknowledgements

[1] D.L. Feder, C.W. Clark and B.I. Schneider, Phys. Rev. A 61, 011601 (2000).
[2] B. Jackson, J.F. McCann and C.S. Adams, Phys. Rev. A 61, 051603(R) (2000).
[3] D.L. Feder, A.A. Svidzinsky, A.L. Fetter and C.W. Clark, Phys. Rev. Lett. 86, 564 (2001).
[4] A. Aftalion and Q. Du, cond-mat/0103299.
[5] M. Tsubota, K. Kasamatsu and M. Ueda, cond-mat/0104523.
[6] D.L. Feder and C.W. Clark, cond-mat/0108019.
[7] B.Y. Rubinstein and L.M. Pismen, Physica D **78**, 1 - 10 (1994)
[8] L.M. Pismen, *Vortices in Nonlinear Fields: From liquid crystals to superfluids, from non-equilibrium patterns to cosmic strings* (Oxford University Press, 1999).
[9] Y. Castin and R. Dum, Eur. Phys. J. D 7, 399 (1999).
[10] P.O. Fedichev and G.V. Shlyapnikov, Phys. Rev. A 60, R1779 (1999).
[11] A.A. Svidzinsky and A.L. Fetter, Phys. Rev. A 62, 063617 (2000).
[12] M. Guilleumas and R. Graham, Phys. Rev. A 64, 033607 (2001).
[13] A.A. Svidzinsky and A.L. Fetter, cond-mat/0102003.
[14] M Kraemer, L. Pitaevskii, S. Stringari and F. Zambelli, cond-mat/0106524.
[15] M.R. Matthews et al., Phys. Rev. Lett. 83, 2498 (1999).
[16] B.P. Anderson, P.C. Haljan, C.E. Wieman and E.A. Cornell, cond-mat/0005368.
[17] K.W. Madison, F. Chevy, W. Wohlleben and J. Dalibard, Phys. Rev. Lett. 84, 806 (2000).
[18] K.W. Madison, F. Chevy, V. Bretin and J. Dalibard, ibid. 86, 4443 (2001).
[19] J.R. Abo-Shaeer, C. Raman, J.M. Vogels, and W. Ketterle, Science 292, 476 (2001).
[20] C. Raman et al., cond-mat/0106235.
[21] P.C. Haljan, I. Coddington, P. Engels, and E.A. Cornell, cond-mat/0106362.
[22] E. Hodby et al., cond-mat/0101292.
[23] A.E. Muryshev and P.O. Fedichev, cond-mat/0106462.
[24] J.R. Anglin, cond-mat/0106619.
[25] J.J. Garcia-Ripoll and V.M Perez-Garcia, Phys. Rev. A 63, 041603(R) (2001) ; A. Aftalion and T. Riviere, cond-mat/0105208.
[26] D. S. Petrov, M. Holzmann and G. V. Shlyapnikov, Phys. Rev. Lett. 84, 2551 (2000).
[27] E. Lundh, C.J. Pethick and H. Smith, Phys. Rev. A 55, 2126 (1996); see also F. Dalfovo, L.P. Pitaevskii and S. Stringari, ibid. 54, 4213 (1996).
[28] Th. Busch and J.R. Anglin, Phys. Rev. Lett. 84, 2298 (2000); ibid. 87, 010401 (2001).
[29] U. Al Khawaja, C.J. Pethick, and H. Smith, Phys. Rev. A 60, 1507 (1999).
[30] I.S. Gradshteyn and I.M. Rhyzhik, *Table of Integrals, Series, and Products*, 5th ed. (Academic, 1994), p. 752, #6.672–4.
[31] ibid., p. 1034 #8.820-2 and p. 1065 #9.100.
[32] ibid., p. 1036, #8.822-2.
[33] Wolfram Research Inc., *Mathematica*, 4th ed., (Champaign, IL, 1999).
[34] J.R. Abo-Shaeer, private communication.